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Improved dimension reduction method (DRM) in uncertainty analysis using kriging interpolation †

Junho Won[1], Changhyun Choi[1] and Jooho Choi[2,*]

[1]Department of Mechanical and Aerospace Engineering, Korea Aerospace University, Goyang, Gyeonggi 412-791, Korea
[2]School of Mechanical and Aerospace Engineering, Korea Aerospace University, Goyang, Gyeonggi 412-791, Korea

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Abstract

Uncertainty or reliability analysis is to investigate the stochastic behavior of response variables due to the randomness of input parameters, and evaluate the probabilistic values of the responses against the failure, which is known as reliability. While the major research for decades has been made on the most probable point (MPP) search methods, the dimension reduction method (DRM) has recently emerged as a new alternative in this field due to its sensitivity-free nature and efficiency. In the recent implementation of the DRM, however, the method was found to have some drawbacks which counteract its efficiency. It can be inaccurate for strong nonlinear response and is numerically instable when calculating integration points. In this study, the response function is approximated by the Kriging interpolation technique, which is known to be more accurate for nonlinear functions. The integration is carried out with this metamodel to prevent the numerical instability while improving the accuracy. The Kriging based DRM is applied and compared with the other methods in a number of mathematical examples. Effectiveness and accuracy of this method are discussed in comparison with the other existing methods.

Keywords: Sensitivity-free approach; Uncertainty analysis; Dimension reduction method; Kriging interpolation

1. Introduction

In recent years, uncertainty analysis has gained attention for more accurate evaluation of design quality, which is to quantify how much the response or performance is affected due to the uncertainty of input variables. Upon uncertainty quantification, reliability analysis can be carried out in which the failure probability is calculated, which is of great importance to the design engineer. Among the various tools developed in this field for the last decades, the most popular methods are MPP based methods, which include FORM, SORM, and so on [1]. The methods, however, have some disadvantages in that first-order sensitivities of the system response are required. On the other hand, a new efficient method called univariate dimension reduction method (DRM) was recently proposed by Rahman [2], which is to calculate statistical moments by transforming a multi-dimensional response function into multiple one-dimensional functions. Numerical integration based on the appropriate quadrature is then carried out for each single variable function at a set of integration points. Once the moments are obtained, statistical PDF shape is identified by using the Pearson system, from which the probability can be calculated. This method is more tractable for its sensitivity-free nature and providing the response PDF in a few numbers of analyses. In the recent implementation of the DRM by the authors, however, it was found that the method also has some drawbacks which counteract its efficiency. The method can be inaccurate for strong nonlinear response and is numerically instable when solving a system of linear equa-

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*Corresponding author. Tel.: +82 2 300 0117, Fax.: +82 2 3158 2191
E-mail address: jhchoi@kau.ac.kr

tions in order to determine integration points. In order to circumvent this problem, the response function is approximated by employing the Kriging interpolation method [3], which is known to be capable of more accurately modeling nonlinear responses. The integration is then carried out with this meta-model to prevent the numerical instability while improving the accuracy. Once the first four statistical moments are determined by the DRM, the Pearson system [4] is applied to construct the PDF of the response function. The reliability or failure probability can be calculated by using this PDF. The Kriging based DRM is applied and compared with the other methods in a number of mathematical examples. Effectiveness and accuracy of this method are discussed in comparison with the other traditional methods. Finally, a reliability based design optimization is conducted for a simple mathematical problem to illustrate the effectiveness of the proposed method.

2. Review of univariate dimension reduction method

Recently, the univariate dimension reduction method (DRM) was developed by Rahman and Xu [2] and was applied later to the RBDO problems by Lee, et al. [5]. The idea is to efficiently compute statistical moments of the response function, from which the PDF curve can be constructed by using the Pearson system. CDF is then obtained by the integration of the PDF curve, from which the probability of the response function can be determined. During the moments calculation, the response function of N dimension is approximated into N one-dimensional functions by using the idea of additive decomposition, where N is the number of random variables. Then the original N-dimensional integral of the statistical moment is replaced by a number of one-dimensional integrals, which is much cheaper to compute. The idea is briefly reviewed here. The DRM is to evaluate the following statistical moment defined as

$$\mathrm{E}\left[G^m(\mathbf{X})\right]=\iint\cdots\int G^m(x_1,x_2,...,x_N)\, f_X(x_1,x_2,...,x_N)dx_1dx_2\cdots dx_N \quad (1)$$

where m represents the order of the moment being calculated, G is the response of the system, and (x_1, x_2,…, x_N) are the random input variables. Additive decomposition concept is applied to approximate the response function as follows.

$$\begin{aligned} G(X_1,...,X_N) &\cong \hat{G}(X_1,...,X_N) \\ &= \sum_{j=1}^{N} G(\mu_1,...,\mu_{j-1},X_j,\mu_{j+1},...,\mu_N) \\ &\quad -(N-1)G(\mu_1,...,\mu_N) \end{aligned} \quad (2)$$

Then Eq. (1) becomes multiple one-dimensional integrations that are easier to compute. After replacing G in Eq. (1) by Eq. (2), one-dimensional integral comes up in the resulting expression as follows.

$$\int_{-\infty}^{\infty} G^i(\mu_1,...,\mu_{j-1},x_j,\mu_{j+1},...,\mu_N)\cdot f_{X_j}(x_j)\cdot dx_j \quad (3)$$

where i is an arbitrary integer that is smaller than m and f_{Xj} (x_j) is the marginal probability density of X_j, which can be calculated from the known joint density of $\mathbf{X}$. To compute this integral, a moment-based quadrature rule is introduced, which will allow numerical integration of the function. The raw statistical moments of the random input variables are used to calculate the integration points and weights required for the integration. By applying to a moment-consistent integration rule [2], the following linear system can be constructed.

$$\begin{bmatrix} \mu_{j,n-1} & -\mu_{j,n-2} & \mu_{j,n-3} & \cdots & (-1)^{n-1}\mu_{j,0} \\ \mu_{j,n} & -\mu_{j,n-1} & \mu_{j,n-2} & \cdots & (-1)^{n-1}\mu_{j,1} \\ \mu_{j,n+1} & -\mu_{j,n} & \mu_{j,n-1} & \cdots & (-1)^{n-1}\mu_{j,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_{j,2n-2} & -\mu_{j,2n-3} & \mu_{j,2n-4} & \cdots & (-1)^{n-1}\mu_{j,n-1} \end{bmatrix} \begin{bmatrix} r_{j,1} \\ r_{j,2} \\ r_{j,3} \\ \vdots \\ r_{j,n} \end{bmatrix} = \begin{bmatrix} \mu_{j,n} \\ \mu_{j,n+1} \\ \mu_{j,n+2} \\ \vdots \\ \mu_{j,2n-1} \end{bmatrix} \quad (4)$$

where $\mu_{j,i}$ represents the i^{th} raw moment for the j^{th} random variable and n is the number of integration points. The solution of the linear system becomes the coefficients of the following nonlinear equation whose solution becomes the integration points.

$$x_j^n - r_{j,1}x_j^{n-1} + r_{j,2}x_j^{n-2} - \cdots + (-1)^n r_{j,n} = 0 \quad (5)$$

Once a set of integration points has been determined, numerical integration can be conducted by obtaining the corresponding weights as follows.

$$w_{j,k} = \frac{\int_{-\infty}^{+\infty} \prod_{l=1,l\neq k}^{n}(x_j - x_{j,l}) f_{X_j}(x_j)dx_j}{\prod_{l=1,l\neq k}^{n}(x_{j,k}-x_{j,l})} = \frac{\sum_{l=0}^{n-1}(-1)^l \mu_{j,n-l-1} q_{j,kl}}{\prod_{l=1,l\neq k}^{n}(x_{j,k}-x_{j,l})}$$

$$q_{j,k0} = 1$$
$$q_{j,kl} = r_{j,l} - x_{j,k} q_{j,k(l-1)} \tag{6}$$

where $w_{i,k}$ represents the weight at the k^{th} integration point for the j^{th} random variable. Then the integral of Eq. (3) becomes the following algebraic operation:

$$\sum_{k=1}^{n} w_{j,k} G^i (\mu_1, ..., \mu_{j-1}, x_{j,k}, \mu_{j+1}, ..., \mu_N) \tag{7}$$

Note that this method requires $(n-1)N+1$ number of analyses for the moments calculation if the nominal point is used in common at each one-dimensional integration. In case of moderate nonlinearity of the response function, one can get fairly accurate results with n=5 and under. More accuracy or higher nonlinearity can be accommodated by further increasing the number of integration points n. As will be shown later, however, the result for some problems is not as good as expected, nor is it converged despite n being increased. Furthermore, singularity of matrix in Eq. (4) is often observed at higher n, which leads to the breakdown of the process. To circumvent this problem, the response function is approximated by employing the Kriging interpolation method [3], which is known to be capable of more accurately modeling nonlinear responses. The integration is then carried out with this meta-model to prevent numerical instability while improving the accuracy.

3. Approximating the response function via Kriging interpolation technique

When constructing approximation models from a set of sample data, response surface techniques based on polynomial regression have been in the majority for a long time due to its simplicity. This method, however, has its origin from the physical experiments in which the results can vary from each other at the same design point. Therefore, a number of new model building strategies have recently been proposed, which are especially suited for a deterministic computer response that always produces the same result at a design point. Unlike the traditional regression model, it is more advantageous for this model to predict exactly the calculated responses at the sample points, and control the flexibility or smoothness of the function at the untried points. There have been two approaches in this direction. One is the moving least-squares (MLS) method proposed by Lancaster and Salkauskas [6], and its improved version referred to as the stepwise MLS (SMLS) method. The idea is to start with a weighted least squares formulation for an arbitrary fixed point, and then move this point over the entire parameter domain, where a weighted least squares fit is computed and evaluated for each point individually. In the SMLS method, the stepwise regression scheme is further employed by adaptively selecting basis functions in order to achieve the best approximation. So, compared to the MLS, the SMLS improves numerical accuracy. The MLS method has, however, some drawbacks in that it is not easy to find the proper basis functions which contribute the most to the accuracy of the approximation. Besides, the parameter in the weighting function that controls the degree of interpolation/approximation adversely affects the degree of smoothness, i.e., if the parameter is selected to get closer to the interpolated function, one tends to get poor smoothness of the function, and vice versa. The other approach is the Kriging interpolation or DACE method proposed by Sacks et al., which is employed in this study. In the Kriging method, there is a similar function named correlation and its associated parameter. Unlike the MLS method, the parameter in the Kriging method is used only to control the degree of smoothness because it always interpolates the response at the sample points. Besides, the Kriging method does not require one to choose basis functions as used in the MLS. The algorithm is briefly reviewed as follows.

Given a set of n sample points $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n]'$ with $\mathbf{x} \in R^N$ and the responses $\mathbf{Y} = [y_1, y_2, ..., y_n]'$. Let us express true y as a summed realization of a regression model and a random function

$$y(\mathbf{x}) = \mathbf{f}(\mathbf{x})' \beta + z(\mathbf{x}) \tag{8}$$

where the number of trial functions $\mathbf{f}(\mathbf{x})$ and regression parameters β is k. Random process z($\mathbf{x}$) is assumed a statistical error, having a normal distribution with zero mean and variance σ^2, i.e.,

$$E\left[z(\mathbf{x}_i) z(\mathbf{x}_j)\right] = E\left[z_i z_j\right] = \sigma^2 R(\mathbf{x}_i, \mathbf{x}_j) = \sigma^2 R_{ij} \tag{9}$$

where σ is the standard deviation of the response, and R_{ij} is the correlation matrix in which the values are determined by the correlation function. At untried arbitrary point **x**, one can define

$$E\left[z\left(\mathbf{x}_i\right)z\left(\mathbf{x}\right)\right]=E\left[\mathbf{Z}z\left(\mathbf{x}\right)\right] =\sigma^2 R\left(\mathbf{x}_i,\mathbf{x}\right)=\sigma^2 r_i\left(\mathbf{x}\right) \tag{10}$$

where $r_i\left(\mathbf{x}\right)=R\left(\theta,\mathbf{x}_i,\mathbf{x}\right)$.

The surrogate y is expressed by using a linear predictor for predicting at untried point **x**:

$$\hat{y}\left(\mathbf{x}\right)=\mathbf{c}\left(\mathbf{x}\right)'\mathbf{Y}=\mathbf{c}\left(\mathbf{x}\right)'\left[\mathbf{F}\beta+\mathbf{Z}\right] \tag{11}$$

where **c**(**x**) is a set of functions interpolating the given **Y** data at the current **x**. Matrix **F** with dimension $n\times k$ is defined from $F_{ij}=f_j(\mathbf{x}_i)$, and the error vector **Z** is from $Z_i = z(\mathbf{x}_i)$. Then the error is defined as

$$\begin{aligned}\varepsilon\left(\mathbf{x}\right)&=\hat{y}\left(\mathbf{x}\right)-y\left(\mathbf{x}\right)\\ &=\mathbf{c}\left(\mathbf{x}\right)'\left[\mathbf{F}\beta+\mathbf{Z}\right]-\left[\mathbf{f}\left(\mathbf{x}\right)'\beta+z\left(\mathbf{x}\right)\right]\\ &=\mathbf{c}\left(\mathbf{x}\right)'\mathbf{Z}-z\left(\mathbf{x}\right)+\left[\mathbf{F}'\mathbf{c}\left(\mathbf{x}\right)-\mathbf{f}\left(\mathbf{x}\right)\right]'\beta\end{aligned} \tag{12}$$

To keep the predictor unbiased, we demand

$$\mathbf{F}'\mathbf{c}\left(\mathbf{x}\right)-\mathbf{f}\left(\mathbf{x}\right)=\mathbf{0} \quad \text{or} \quad \mathbf{F}'\mathbf{c}\left(\mathbf{x}\right)=\mathbf{f}\left(\mathbf{x}\right) \tag{13}$$

This is a kind of constraint for the weight functions. In a simple case where **f**(**x**) = [1], and $\mathbf{F}=\left[1,1,...,1\right]'$ which is $N\times1$ vector,

$$\mathbf{F}'\mathbf{c}\left(\mathbf{x}\right)=\sum_{i=1}^{N}c^i\left(\mathbf{x}\right)=1 \tag{14}$$

which means that the sum of the weight or the coefficient functions should be unity. Under this condition, MSE becomes

$$MSE=E\left[\varepsilon^2\right]=\sigma^2\left(1-2\mathbf{c}'\mathbf{r}+\mathbf{c}'\mathbf{R}\mathbf{c}\right) \tag{15}$$

c(**x**) is determined such that *MSE* is minimized subject to the constraint. Then the surrogate y becomes

$$\hat{y}\left(\mathbf{x}\right)=\mathbf{f}'\beta^*+\mathbf{r}'\mathbf{R}^{-1}\left(\mathbf{Y}-\mathbf{F}\beta^*\right) \tag{16}$$

where $\beta^*=\left(\mathbf{F}'\mathbf{R}^{-1}\mathbf{F}\right)^{-1}\left(\mathbf{F}'\mathbf{R}^{-1}\mathbf{Y}\right)$, which is $\left(k\times1\right)$ vector. Minimum *MSE* becomes

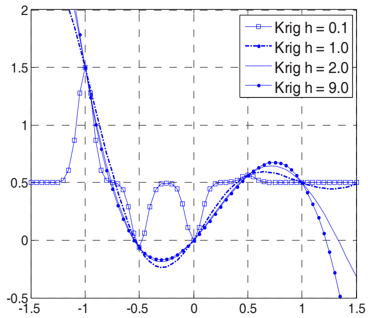


Fig. 1. Kriging approximated function.

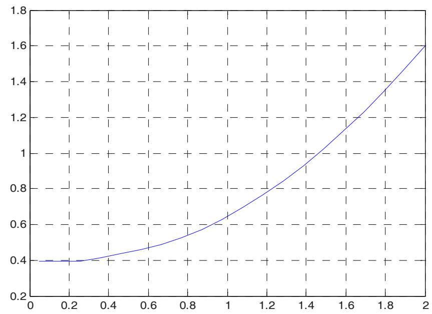


Fig. 2. MLE function.

$$MSE=\sigma^2\left\{1-\left[\mathbf{f}'\,\mathbf{r}'\right]\begin{bmatrix}0 & \mathbf{F}'\\ \mathbf{F} & \mathbf{R}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{f}'\\ \mathbf{r}'\end{bmatrix}\right\} \tag{17}$$

Note here that **f**, **r** are functions of the current point **x** whereas **F**, **R**, **Y** and β^* are constants. Unlike the case of regular regression, the MSE given by Eq. (17) is a function of **x**, which can be plotted.

In the Kriging method, there are several types of the correlation function. In this study, the Gaussian type function is used as follows.

$$R\left(\mathbf{x}_i,\mathbf{x}_j\right)=\exp\left\{-\left(\frac{d}{h}\right)^2\right\},\quad d=\left\|\mathbf{x}_i-\mathbf{x}_j\right\| \tag{18}$$

where d is distance between the two points, and h is an arbitrary correlation parameter which affects the smoothness of the model. In most Kriging studies, h is determined by the method of maximum likelihood estimate (MLE). According to Etman [7] and Sasena

[8], however, the MLE method is not only computationally expensive, which requires an additional optimization process, but also the quality of the obtained parameter is questionable.

In this study, the Kriging approximation is applied to one dimensional function with 3 or 5 numbers of points. As a feasibility study, consider a simple test function

$$y = -1.5x^3 + x^2 + x \tag{19}$$

Using 5 sample points given by $x_i = \{-1, -.5, 0, .5, 1\}$ and their responses, an approximate function is plotted for four h's 0.1, 1.0, 2.0 and 10.0 in Fig. 1. It is observed that the curve becomes smoother and gets closer to the original function as h is increased. In Fig. 2, the MLE function is also plotted in terms of h, in which the optimum solution for h is found at the lower limit value. It is obvious that the solution is wrong, which yields a poor fit as shown in Fig. 1. This agrees with the findings [7, 8] that the MLE method can be doubtful. If, on the other hand, h is increased constantly, a singularity problem for R matrix is encountered, in which all the row values become similar and get closer to the unit value. Based on these observations, a new and simple criterion for proper choice of h is proposed, i.e., large h is chosen such that the curve is sufficiently smooth but not so large as to cause singularity of the R matrix.

Remembering that in the R matrix the maximum value is unity at the diagonal location while it decreases as the location gets off-diagonal, the level of singularity is determined by the ratio d_{max}/h, where d_{max} is the maximum distance among the design points. After a number of trials, the proper value for this ratio is found to be 0.5, which produces the minimum value in the matrix at $\exp(-0.5^2)$ =0.779. In this test problem, this corresponds to h=4.

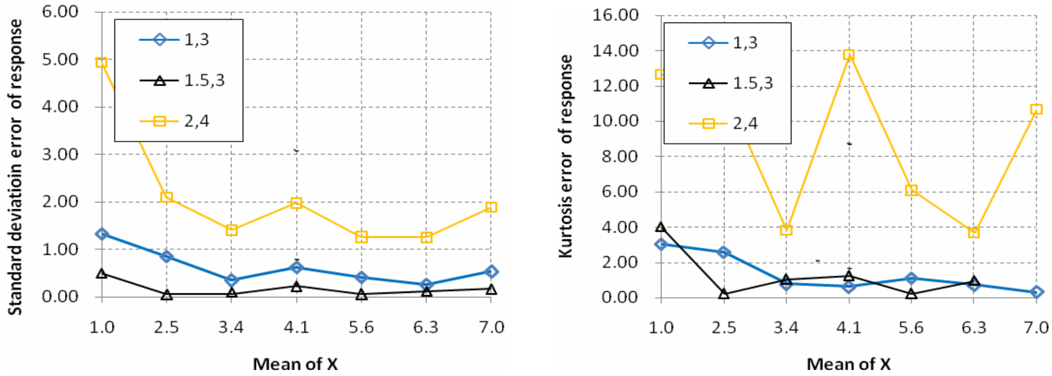
4. Axial DOE study

In the implementation of the DRM, the one-dimensional function $\tilde{G}(x_i)$ given in Eq. (2) is approximated by the Kriging method based on the response values at the sampling points defined axially along x_i. If the nonlinearity of G or input variances are small, 3 sampling points are fine including the nominal point. Otherwise, 5 points are necessary. This amounts to total $2N$+1 and $4N$+1 number of sample points, respectively.

Selecting proper sample point locations to best approximate the response is another important task. Since there is no absolute criterion on the best location of the points, they are selected in an ad hoc way based on the number of trials. To this end, two problems are studied to determine 5 sample point locations. First is a single variable problem, in which the mean values are varied under a fixed value of standard deviation as follows.

$$G(X) = \frac{X^2(2+\sin(2X))}{4} \tag{20}$$

where $X \to N(\mu, 0.3^2)$, μ =1, 2.5, 3.4, 4.1, 5.6, 6.3, 7. In Fig. 3, errors of standard deviation and kurtosis of G against MCS simulation with 1E6 numbers are



(a) Standard deviation error (b) Kurtosis error

Fig. 3. Errors of standard deviation and kurtosis of Eq. (20).

plotted for a number of sample point trials. In the figure, the two numbers denote sigma values to the right of the mean point, e.g., if they are (1, 3), then the 5 points are $-3\sigma, -\sigma, 0, \sigma, 3\sigma$ with respect to the mean value. The accuracy is slightly better at (1.5, 3). But it fails at μ =7. Next is a two variables problem, in which the standard deviations of the two input variables are varied together under a fixed mean values.

$$Y(X_1, X_2) = \frac{1}{1 + X_1^4 + 2X_2^2 + 5X_2^4} \tag{21}$$

where $X_j \to N(0, \sigma^2)$, $j = 1,2$. In Fig. 4, errors of mean and standard deviation of G against MCS are plotted. In this case, (1, 3) is much better than the other two. The conclusion from these two is that (1, 3) is chosen in this study, for the 3 points, one just chooses a mean value intuitively between (1, 3), which yields $-2\sigma, 0, 2\sigma$ with respect to the mean value. This corresponds to (2.3%, 50%, 97.7%) in the case of the normal distribution. In the 5 point case, this is (0.13%, 15.9%, 50%, 84.1%, 99.87%).

In the case of non-normal PDF, corresponding levels can be calculated from the CDF values to match these values. Once the approximate function is constructed by the Kriging method, one can perform numerical integration of Eq. (3) by using a standard quadrature rule at as many integration points as possible because the function value is obtained not from the original but from the surrogate model. In this study, adaptive Simpson quadrature is used. Based on this approach, the statistical moments can be successfully computed without singularity problem while improving the accuracy.

5. Pearson system for PDF construction

Once the four statistical moments (mean, standard deviation, skewness and kurtosis) are obtained, the Pearson System [4] can be used to construct the PDF of the response. The detail expression of the PDF can be achieved by solving the differential equation as

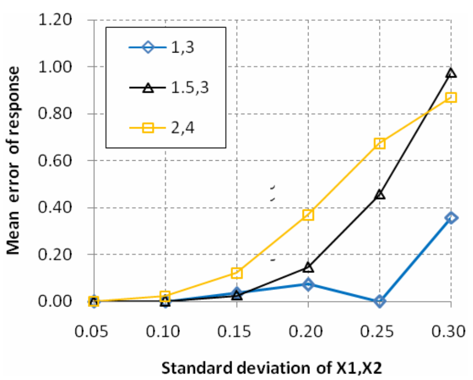
$$\frac{1}{p}\frac{dp}{dx} = -\frac{a + x}{c_0 + c_1 x + c_2 x^2} \tag{22}$$

where a, c_0, c_1 and c_2 are four coefficients determined by the first four moments and expressed as

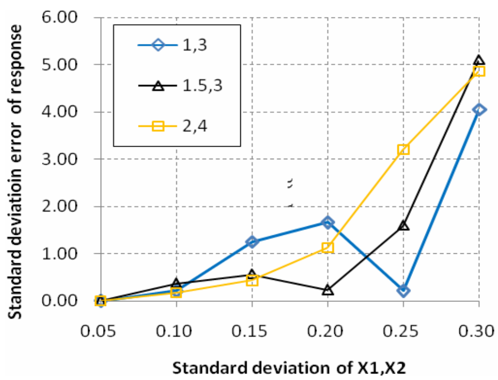
$$c_0 = (4\beta_2 - 3\beta_1)(10\beta_2 - 12\beta_1 - 18)^{-1}\mu_2$$
$$c_1 = a = \sqrt{\beta_1}(\beta_2 + 3)(10\beta_2 - 12\beta_1 - 18)^{-1}\sqrt{\mu_2}$$
$$c_2 = (2\beta_2 - 3\beta_1 - 6)(10\beta_2 - 12\beta_1 - 18)^{-1}$$

where β_1 is the square of skewness, β_2 is the kurtosis, and μ_2 is the variation.

Generally, there are seven distribution types in the Pearson System based on the four coefficients, and among some types, subtypes are present. Normally, PDF can be successfully constructed based on the first four moments. However, the Pearson system can easily fail to construct the PDF, especially when the locations of statistical moments in the Pearson curve simultaneously approach the limit curves of several distribution types as shown in Fig. 5. The horizontal axis is for the square of skewness and vertical axis is for the kurtosis.



(a) Mean error



(b) Standard deviation error

Fig. 4. Errors of mean and standard deviation of Eq. (21).

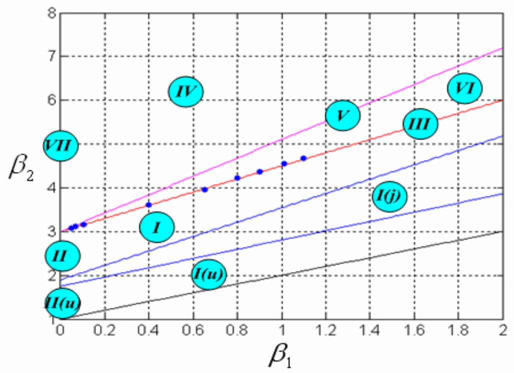


Fig. 5. Pearson curve.

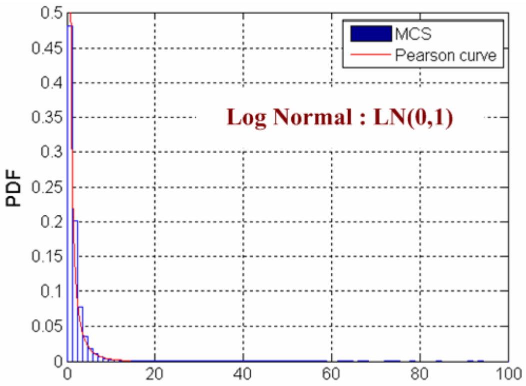


Fig. 6. Frequency diagram of MCS & PDF of pearson system for $LN(0,1)$.

In order to verify the Pearson system, several existing distributions are tested, which is to generate 1E6 random data of each distribution, compute first four moments, construct PDF and CDF curve and compute the difference from the original CDF as an accuracy measure. As shown in Table 1, the difference is quite satisfactory except in three cases. In the case of $LN(0,1)$, the difference is 8.46E-2 which is not small. As shown in Fig. 6, however, the PDF is one of the extreme cases, and is not likely to occur. Three other cases failed to produce PDF. Tests are also conducted for arbitrary distributions made by combining two normal distributions $X \sim N(10,3)$ and $Y \sim N(3,2)$, the results of which are given in Table 2. Again, poor accuracies are found in some cases. As mentioned above, this PDF may also rarely occur in normal problems as shown in Fig. 7 for the Y^4 case. In Fig. 5, failure cases are marked as blue dots, which are at type III which is the boundary of type VI and I.

The trouble lies in the calculation of coefficients of a specific distribution type, which results in a

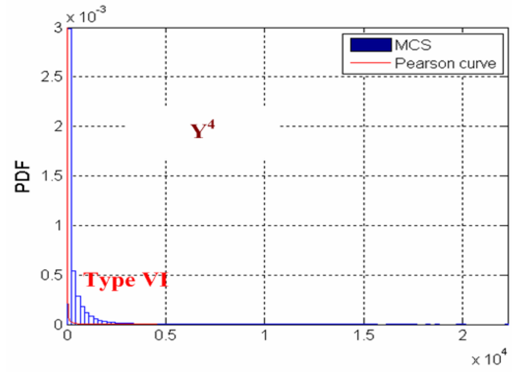


Fig. 7. Frequency diagram of MCS & PDF of pearson system for Y^4.

numerical instability. In this case, two PDFs are generated by fixing the first three statistical moments, and incrementally adjusting the original kurtosis by slightly increasing and decreasing the value until two PDFs are successfully constructed. If the two PDFs are close to each other within 0.001, either one is used. Otherwise, a Kriging interpolated curve is generated from the two PDFs. This method of adjustment was originally studied by Youn et al. [18]. In Fig. 8, PDF after the adjustment is drawn for the case of GEV(-1, 100), which results in excellent agreement with the original PDF.

6. Examples of uncertainty analysis

Four examples are studied to verify the developed method. First two are from Rahman [2].

$$Y(X_1, X_2) = \frac{1}{1 + X_1^4 + 2X_2^2 + 5X_2^4} \quad \& \\ Y(X_1, X_2) = \exp\left(-\frac{1}{1 + 100X_1^2 + 2X_2^2 + X_1^2 X_2^2} \right) \tag{23}$$

where $X_j \rightarrow N(0, \sigma^2), \quad j = 1, 2$. Note that the first one is already used for the axial DOE study. As can be seen in Fig. 9 the two functions are heavily dependent on either variable, i.e., X_2 and X_1 respectively. Kriging based DRM (named KDRM hereafter) is conducted to explore the trend of output standard deviation with the increase of two input standard deviations. In Fig. 10 are the results of the study in which the results by the conventional DRM are compared along with those by the Monte Carlo simulation

Table 1. Validation of existing distribution.

	Distribution Type		Mean	Standard Deviation	Skewness	Kurtosis	Pearson type	$MAX\left[\left\lvert F_{MCS}(x) - F_{pearson}(x)\right\rvert\right]$
1	Normal	N(10,3)	1.00E+01	3.00E+00	2.15E-03	3.00E+00	Nomal	4.62E-04
2		N(3,2)	3.00E+00	2.00E+00	-6.88E-04	3.00E+00	Nomal	5.85E-04
3	Chi-square	X2(1)	1.00E+00	1.41E+00	2.82E+00	1.48E+01	Type 6	Fail
4		X2(9)	9.00E+00	4.24E+00	9.39E-01	4.32E+00	Type I	3.20E-04
5		X2(45)	4.50E+01	9.48E+00	4.18E-01	3.26E+00	Type I	2.01E-04
6	Exponential	EP(1)	1.00E+00	1.00E+00	2.00E+00	8.99E+00	Type 6	Fail
7		EP(2)	2.00E+00	2.00E+00	1.99E+00	8.93E+00	Type I	7.78E-03
8		EP(45)	4.50E+01	4.50E+01	2.01E+00	9.05E+00	Type I	2.65E-02
9	Extreme Value	EV(1,1)	4.25E-01	1.28E+00	-1.14E+00	5.42E+00	Type 6	3.13E-03
10		EV(2,1)	1.42E+00	1.28E+00	-1.13E+00	5.38E+00	Type 6	3.22E-03
11		EV(2,2)	8.43E-01	2.57E+00	-1.14E+00	5.42E+00	Type 6	3.89E-03
12		EV(2,5)	-8.96E-01	6.41E+00	-1.13E+00	5.38E+00	Type 6	3.02E-03
13	F	F(10,100)	1.02E+00	4.84E-01	1.05E+00	4.78E+00	Type 6	7.35E-04
14		F(100, 10)	1.25E+00	7.51E-01	3.33E+00	3.32E+01	Type 4	2.58E-03
15		F(100,100)	1.02E+00	2.07E-01	6.25E-01	3.72E+00	Type 4	3.25E-04
16	Gamma	G(1,1)	1.00E+00	9.99E-01	2.00E+00	9.00E+00	Type I	2.18E-02
17		G(4,1)	4.00E+00	2.00E+00	9.97E-01	4.49E+00	Type I	4.42E-03
18		G(5,0.3)	1.50E+00	6.71E-01	8.91E-01	4.18E+00	Type 3	3.18E-04
19	Generalized Extreme Value	GEV(0.1,100,1)	6.96E+01	1.49E+02	1.92E+00	1.12E+01	Type 6	2.94E-03
20		GEV(-0.5,100,1)	2.38E+01	9.26E+01	-6.33E-01	3.25E+00	Type I	3.57E-03
21		GEV(-1,100,1)	1.05E+00	9.99E+01	-1.99E+00	8.91E+00	Type 6	Fail
22	Generalized Pareto	GPR(0.1,100,2)	1.13E+02	1.24E+02	2.79E+00	1.72E+01	Type 6	3.52E-04
23		GPR(-1.0,100,2)	5.20E+01	2.89E+01	-6.16E-05	1.80E+00	Type I	3.52E-04
24		GPR(-1.1,100,2)	4.96E+01	2.66E+01	-8.32E-02	1.79E+00	Type I	5.86E-04
25	Weibull	W(2, 2)	1.77E+00	9.26E-01	6.30E-01	3.25E+00	Type I	3.46E-03
26		W(2,10)	1.90E+00	2.29E-01	-6.39E-01	3.57E+00	Type I	3.31E-03
27	Student-t	T(5)	-2.59E-03	1.29E+00	2.59E-02	8.42E+00	Type 4	2.82E-04
28		T(9)	3.68E-05	1.13E+00	2.08E-03	4.17E+00	Type 4	6.60E-04
29		T(45)	-7.26E-04	1.02E+00	3.98E-03	3.14E+00	Type 4	4.90E-04
30	Rayleigh	RAYL(1)	1.25E+00	6.56E-01	6.32E-01	3.25E+00	Type I	3.13E-03
31		RAYL(10)	1.25E+01	6.56E+00	6.33E-01	3.25E+00	Type I	3.02E-03
32	Log Normal	LN(0,0.1)	1.01E+00	1.01E-01	2.99E-01	3.16E+00	Type 6	3.04E-04
33		LN(0, 1)	1.65E+00	2.17E+00	6.05E+00	9.76E+01	Type 6	8.46E-02

with 1E6 numbers. Both $2N+1=5$ (3 points axially) and $4N+1=9$ (5 points axially) points are tried. In the first problem, the KDRM is better than the DRM at 3 points, and is comparable at 5 points. In the second problem, the KDRM is again better than the DRM at 3 points, but is worse than the DRM at 5 points. Third problem is also from Rahman [2], which is

$$Y\left(X_1,X_2,X_3\right)=3X_1^2-X_1X_2+X_1X_3+X_3^3 \tag{24}$$

$X_j \rightarrow \text{Weibull}\left(0.918,\sigma^2\right)$, $j=1,2,3$. As is shown in Fig. 11, accuracies of the two methods are similar. Magnitudes of the four moments are compared in Table 3 for the case $X_j \rightarrow \text{Weibull}\left(0.918,0.210^2\right)$, $j=1,2,3$ from which the agreements of the two methods are found. Last example is from Zhao & Ono [9], which is

$$Y\left(X_1,X_2,X_3\right)=X_1^2X_2^2+2X_3^4 \tag{25}$$

where $X_j \rightarrow \text{Lognormal}\left(1.0,\sigma^2\right)$, $j=1,2,3$. Fig. 12 shows the result. Accuracy of KDRM is poor at 3 points whereas similar at 5 points. Magnitudes of the four moments are compared in Table 4 for the case

Table 2. Validation of arbitrary distribution.

Equation	Pearson type	$MAX\left[\begin{vmatrix} F_{MCS}(x) \\ -F_{pearson}(x) \end{vmatrix}\right]$	Equation	Pearson type	$MAX\left[\begin{vmatrix} F_{MCS}(x) \\ -F_{pearson}(x) \end{vmatrix}\right]$	Equation	Pearson type	$MAX\left[\begin{vmatrix} F_{MCS}(x) \\ -F_{pearson}(x) \end{vmatrix}\right]$
X	Normal	4.62E-04	X + Y	Normal	3.22E-04	-X + Y	Nomal	4.470E-04
X^2	Type 1	2.903E-03	$X + Y^2$	Type 1	1.804E-02	$-X + Y^2$	Type 1	1.744E-02
X^3	Type 6	1.001E-03	$X + Y^3$	Type 6	2.427E-01	$-X + Y^3$	Type 6	2.429E-01
X^4	Type 6	3.871E-02	$X^2 + Y$	Type 1	2.198E-03	$-X^2 + Y$	Type 1	2.048E-03
Y	Normal	5.854E-04	$X^2 + Y^2$	Type 1	1.414E-03	$-X^2 + Y^2$	Type 1	3.09E-03
Y^2	Type 1	7.42E-02	$X^2 + Y^3$	Type 6	1.71E-02	$-X^2 + Y^3$	Type 4	5.28E-02
Y^3	Type 6	2.47E-01	$X^3 + Y^1$	Type 6	1.65E-03	$-X^3 + Y^1$	Type 6	1.77E-03
Y^4	Type 6	6.93E-01	$X^3 + Y^2$	Type 6	1.56E-03	$-X^3 + Y^2$	Type 6	1.91E-03
			$X^3 + Y^3$	Type 6	2.19E-03	$-X^3 + Y^3$	Type 6	3.65E-03

Table 3. $Y=1/(1+X_1^4+2X_2^2+5X_2^4)$.

	MCS	DRM	Error %	KDR	Error %
Mean	3.5551	3.5553	0.01	3.5557	0.02
STD	1.3724	1.3708	0.12	1.3721	0.02
Skew	0.406	0.3016	25.71	0.407	0.25
Kurt	3.0346	2.9576	2.54	3.0412	0.22

Table 4. $Y=Exp(-1/(1+100X_1^2+2X_2^2+X_1^2X_2^2))$.

	MCS	DRM	Error %	KDR	Error %
Mean	3.1428	3.143	0.01	3.1432	0.01
STD	0.9296	0.9277	0.2	0.9282	0.15
Skew	1.1539	1.1458	0.7	1.1434	0.91
Kurt	5.6388	5.6507	0.21	5.6124	0.47

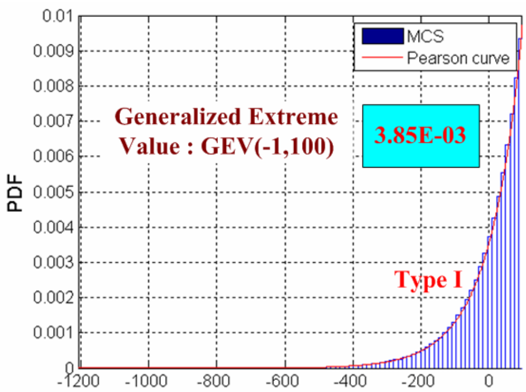
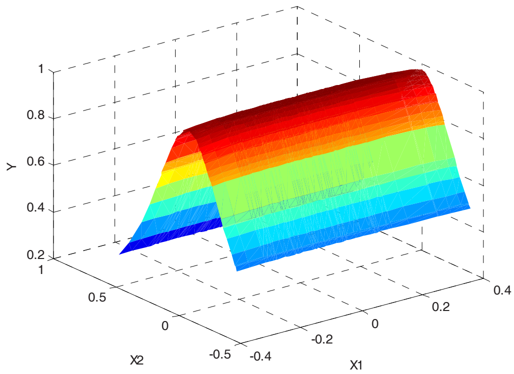
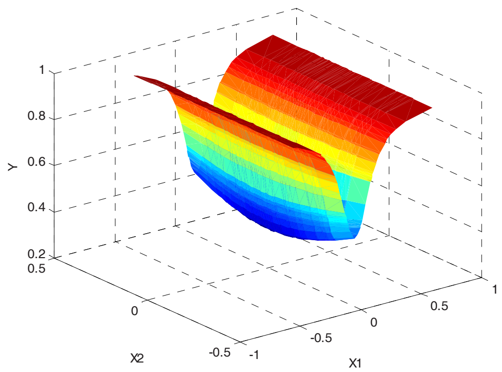


Fig. 8. Frequency Diagram of MCS & PDF of Pearson System for *GEV*(-1,100).



(a) $Y=1/(1+X_1^4+2X_2^2+5X_2^4)$



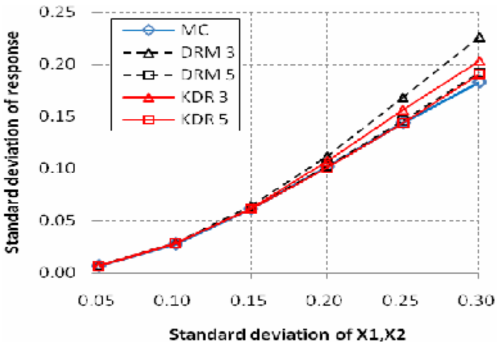
(b) $Y=EXP(-1/(1+100X_1^2+2X_2^2+X_1^2X_2^2))$

Fig. 9. Shape of the two functions in Eq. (23).

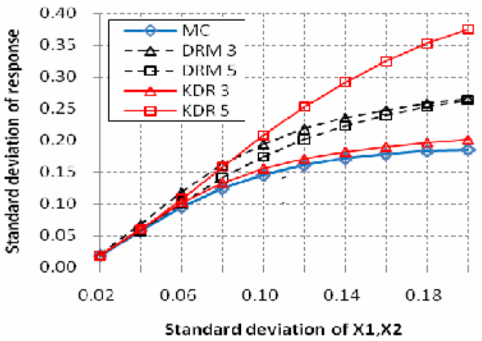
$X_j \rightarrow \text{Lognormal}\left(1.0, 0.1^2\right)$, $j = 1,2,3$ from which the agreements of the two methods are found too.

7. Discussions and conclusions

In the recent study of reliability analysis, the di-



(a) Standard deviation of first function



(b) Standard deviation of second function

Fig. 10. Standard deviation of the two functions in Eq. (23).

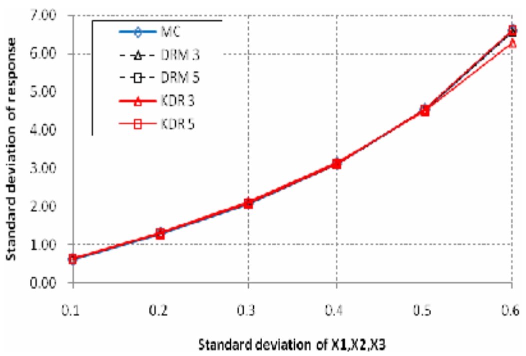


Fig. 11. Standard deviation of the function in Eq. (24).

mension reduction method (DRM) has emerged as a new alternative choice due to its sensitivity-free nature and efficiency. During the implementation of the DRM, however, the method was found to have some drawbacks which counteract its efficiency. It was found to be inaccurate for strong nonlinear responses such as the first and second problems of this paper and can be numerically instable when calculating

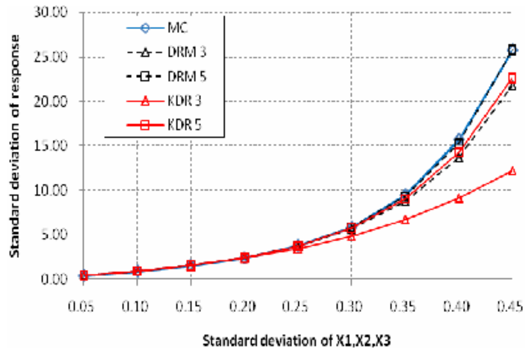


Fig. 12. Standard deviation of the function in Eq. (25).

integration points with more than 5 points. As a remedial approach to this, the Kriging interpolation technique is employed to build a surrogate function, by which the integration is carried out. From the four problems tested, accuracies of the Kriging based DRM and classical DRM are found to be similar each other. Clear advantage of the Kriging based DRM in view of accuracy is not found since the number of problems and trial cases is still too small.

The Kriging based DRM, however, is more tractable because the computation does not fail. Many more test problems along with the parametric variations used in the Kriging method are necessary to be conducted in the future.

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Junho Won received B.S. and M.S. degree in Mechanical and Aerospace Engineering from Korea Aerospace University in 2004 and 2006, respectively. He is currently a doctoral course at the departments of Mechanical and Aerospace Engineering, Korea Aerospace University in Gyeonggi, Korea. His research interests are in the area of reliability analysis, multidisciplinary design optimization, and fatigue analysis.

Changhyun Choi received B.S. and M.S. degree in Mechanical and Aerospace Engineering from Korea Aerospace University in 2006 and 2008, respectively. He is currently researcher at the SFA Engineering Corp. in kyungnam, Korea. . His research interests are in the area of computer control system, high reliable product technology, and factory automation system.

Jooho Choi received a B.S. degree in Mechanical Engineering from Hanyang University in 1981. He then went on to receive his M.S. and Ph.D. degrees from KAIST in 1983 and 1987, respectively. Dr. Choi is currently a Professor at the School of Mechanical and Aerospace Engineering, Korea Aerospace University in Gyeonggi, Korea. He is currently serving as an Editor of the Journal of Mechanical Science &Technology. Dr. Choi's research interests are in the area of reliability analysis, multidisciplinary design optimization, and design optimization using automation system integrated with CAD/CAE.